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Zeta functions of nearly circular domains

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Abstract. We study the zeta functions $\zeta(p;\Omega) = \sum_i \lambda_i^{-p}$ of negative integer powers of eigenvalues of the Laplacian on two-dimensional domains Ω which are close to the unit disc D. For $p = 2, 3, \ldots$, closed-form expressions are obtained which describe $\zeta(p;\Omega)$ when $\Omega \to D$. The technique developed is applied to derive an asymptotic expansion for zeta functions of regular n-sided polygons in the limit $n \to \infty$.

1. Introduction

This paper deals with spectral zeta functions

$$\zeta(p;\Omega) = \sum_i \lambda_i^{-p} \qquad p = 2, 3, \ldots \tag{1}$$

of the Laplacian on a two-dimensional domain Ω with zero boundary condition on $\partial\Omega$. Explicit formulas for such zeta functions, regarded as exact sum rules for the energy levels of quantal systems, are of considerable interest in the theory of quantum billiards [1–7]. The benchmark is the circular billiard, with Ω being the unit disc D. It is one of very few examples where the zeta function can be calculated explicitly for any integer $p \geqslant 2$, being reduced to the well known sum rules [8–13] for negative powers of zeros of Bessel functions. Moreover, much deeper results on structure of $\zeta(p;D)$ in the complex p-plane are known [11, 12, 14]. A q-generalization of these results to discrete circular billiards was obtained in [15, 16]. There are just two more exactly solvable cases where Ω is an equilateral triangle or square [2]. In particular, no explicit formulae are known for zeta functions of regular n-sided polygons with $n \geqslant 5$.

In this paper, we consider two-dimensional domains which in some sense are close to the unit disc, so that the zeta function of D provides first approximation for $\zeta(p;\Omega)$. The goal is to obtain the main correction due to the deviation of Ω from D. As a result, we get approximate formulae for spectral zeta functions of rather general domains which are nearly circular. As a by-product, we also get a set of addition theorems for Bessel functions which seems previously unknown.

A particular case of a nearly circular billiard which deserves special study is when Ω is a regular n-sided polygon P_n with arbitrary number of vertices n. We get asymptotic expansions of the zeta functions $\zeta(p;P_n)$ in the limit $n \to \infty$. They can also be regarded as approximate formulae for finite n. Numerical comparison with the exact values of the zeta functions of an equilateral triangle and a square shows that the approximation is quite

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good even for $n = 3, 4$ and therefore is expected to be very good for higher n where no exact formulae either for eigenvalues or zeta functions are known.

The main idea of our approach is due to a simple observation put forward in [1] and used in [4] for the Aharonov–Bohm circular billiard: if there is a conformal mapping $\omega : D \to \Omega$, the Green function of the Dirichlet problem on Ω is proportional to the corresponding Green function of D with the coefficient $|\mathrm{d}\omega/\mathrm{d}z|^2$. In our case, if Ω is close to D then

$$\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 = 1 + \epsilon(z)$$

where ϵ is small. Evaluating $\zeta(p; \Omega)$ as trace of G_Ω^p and expanding in ϵ yields a perturbation expansion for $\zeta(p; \Omega)$ where the leading term is the zeta function of the unit disc. The next term $\propto \epsilon$ gives the main correction. In the case of a regular n-sided polygon P_n, the situation is slightly more complicated, for the mapping $w : P_n \to D$ has singularities at vertices of P_n. Nevertheless, the general approach still works with some modifications.

Note that our technique and results are restricted to the case of $p = 2, 3, \ldots$. The Green-function method cannot be extended to study the more complicated and interesting case of complex p (the problem of particular importance is the analytic continuation of $\zeta(p; \Omega)$ to $\mathrm{Re}\, p \leqslant 1$).

We describe the main results of the paper in sections 2 and 3, leaving some long derivations to subsequent sections.

2. General domains

Consider the Dirichlet problem

$$-\Delta\Psi = \lambda_i \Psi \qquad \Psi|\partial\Omega = 0 \tag{2}$$

on a simply connected compact domain Ω in the two-dimensional Euclidean plane with a smooth boundary $\partial\Omega$. The zeta function (1) associated with the spectrum of the problem (2) exists for $p > 1$ and can be analytically continued on the whole complex plane as a meromorphic function of p [17]. For integer $p \geqslant 2$, the zeta function is the trace

$$\zeta(p; \Omega) = \mathrm{tr}\, G_\Omega^p = \int_{\omega_i \in \Omega} \prod_{i=1}^{p} \mathrm{d}^2\, \omega_i\, G_\Omega(\omega_i, \omega_{i+1}) \qquad \omega_{p+1} = \omega_1 \tag{3}$$

of the Green function of the problem (2)

$$-\Delta_\omega\, G_\Omega(\omega, \omega') = \delta(\omega, \omega') \qquad G_\Omega|\partial\Omega = 0$$

where Ω is regarded as a domain in the complex ω plane.

Due to the Riemann mapping theorem, there exists a conformal mapping $\omega(z) : D \to \Omega$ of the unit disc D onto Ω such that $\partial\Omega$ is mapped on the unit circle ∂D. The problem (2) on Ω is equivalent to another problem on D

$$-\Delta_z\, \Psi = \lambda_i \left|\frac{\mathrm{d}\omega}{\mathrm{d}z}\right|^2 \Psi \qquad \Psi|\partial D = 0\,.$$

Therefore, one can express the Green function G_Ω through the Green function G_D of the Dirichlet problem on the unit disc:

$$G_\Omega(\omega, \omega') = G_D(z(\omega), z(\omega'))$$

where $z(\omega)$ is the inverse of $\omega(z)$ and G_D is given by Poisson's kernel

$$G_D(z, z') = -\frac{1}{2\pi}\, \ln\left|\frac{z - z'}{1 - z^* z'}\right|\,. \tag{4}$$

Evaluating the trace (3) in terms of the integral over D yields the following representation for integer $p \geqslant 2$:

$$\zeta(p;\Omega) = \operatorname{tr} G_\Omega^p = \operatorname{tr}\left[\left|\frac{\mathrm{d}\omega}{\mathrm{d}z}\right|^2 G_D\right]^p = \int_{z_i \in D} \prod_{i=1}^{p} \mathrm{d}^2 z_i \; \left|\frac{\mathrm{d}\omega}{\mathrm{d}z}(z_i)\right|^2 G_D(z_i, z_{i+1}) \tag{5}$$

where $z_{p+1} = z_1$.

This representation is the starting point of our analysis. In connection with quantum billiards, it first appeared in [1] and later has been used in [4] to study the zeta functions of the Aharonov–Bohm circular billiard.

Now, let Ω be a nearly circular domain. In terms of the mapping ω it means that

$$|\omega(z) - z| \ll |z| \qquad \text{for} \quad z \in D\,.$$

Therefore

$$\left|\frac{\mathrm{d}\omega}{\mathrm{d}z}\right|^2 = 1 + \epsilon(z) \qquad |\epsilon(z)| \ll 1 \qquad \text{for} \quad z \in D \tag{6}$$

and equation (5) can be written as

$$\begin{aligned}\zeta(p;\Omega) &= \operatorname{tr}[(1+\epsilon)G_D]^p = \int_{z_i \in D} \prod_{i=1}^{p} \mathrm{d}^2 z_i \; [1+\epsilon(z_i)]\, G_D(z_i, z_{i+1}) \\ &= \zeta(p; D) + p \; \operatorname{tr}\left(\epsilon G_D^p\right) + \mathrm{O}(\epsilon^2)\,. \end{aligned} \tag{7}$$

The first term is the zeta function of the unit disc.

Let us now recall some known results for the circular billiard. The eigenvalues of the Dirichlet problem on D are zeros j_{mk} of the Bessel functions $J_m(x)$ so that

$$\zeta(p; D) = \sum_{m=-\infty}^{\infty} \zeta_{|m|}(2p) \qquad \zeta_m(2p) = \sum_{k=1}^{\infty} j_{mk}^{-2p}\,. \tag{8}$$

The corresponding eigenfunctions are ($z = r\mathrm{e}^{\mathrm{i}\theta}$)

$$\Psi_{mk}(z) = \left[\sqrt{\pi}\, J'_{|m|}(j_{|m|k})\right]^{-1} \mathrm{e}^{\mathrm{i}m\theta}\, J_{|m|}(j_{|m|k} r)\,. \tag{9}$$

The zeta functions $\zeta_m(2p)$ can be calculated explicitly for any $p = 1, 2, \ldots$ via a recurrence procedure [10]. Lists of them are given in [8, 14]:

$$\zeta_m(2) = \frac{1}{4(m+1)}$$

$$\zeta_m(4) = \frac{1}{2^4(m+1)^2(m+2)}$$

$$\zeta_m(6) = \frac{1}{2^5(m+1)^3(m+2)(m+3)}$$

etc. Substituting these formulae into (8) enables one to evaluate the zeta functions of the circular billiard explicitly:

$$\begin{aligned}\zeta(2; D) &= -\tfrac{5}{32} + \tfrac{1}{8}\,\zeta(2) \\ \zeta(3; D) &= \frac{1}{2^9}\,[25 + 16\zeta(3) - 24\zeta(2)]\end{aligned} \tag{10}$$

etc; $\zeta(p)$ stands for Riemann's zeta function.

We now proceed to evaluate the second term of (7). First, we obtain a representation for $\mathrm{tr}\left(\epsilon\, G_D^p\right)$ in terms of the eigenfunctions (9) of the circular billiard by making use of the spectral decomposition of G_D

$$G_D(z, z') = \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{\Psi_{mk}(z)\Psi_{mk}^*(z')}{j_{|m|k}^2} \tag{11}$$

which yields

$$\mathrm{tr}\left(\epsilon\, G_D^p\right) = \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{1}{j_{|m|k}^{2p}} \langle \Psi_{mk} | \epsilon | \Psi_{mk} \rangle\,.$$

Evaluating the matrix element in polar coordinates by making use of (9) gives the following result.

Theorem 1. For $p = 2, 3, \ldots$

$$\mathrm{tr}\left(\epsilon\, G_D^p\right) = \int_0^1 \epsilon_0(x) g_p(x)\, \mathrm{d}x \tag{12}$$

where

$$\epsilon_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \epsilon\left(\sqrt{x}\mathrm{e}^{\mathrm{i}\theta}\right)\, \mathrm{d}\theta \qquad (z = \sqrt{x}\mathrm{e}^{\mathrm{i}\theta}) \tag{13}$$

and

$$g_p(x) = \sum_{m=0}^{\infty} (2 - \delta_{m0}) S_m(x;\, p) \qquad S_m(x;\, p) = \sum_{k=1}^{\infty} \frac{J_m^2\left(j_{mk}\sqrt{x}\right)}{j_{mk}^{2p}\left[J_m'(j_{mk})\right]^2}\,. \tag{14}$$

The main complications in equation (12) reside in calculating functions g_p. They can be evaluated explicitly for any integer $p \geqslant 2$ by a method similar to the derivation of the Kneser–Sommerfeld expansion [9]. The corresponding technique is described in section 4. For $p = 2, 3$ the results are

$$\begin{aligned} g_2(x) &= -\frac{(1-x)^2}{8x} \ln(1-x) \\ g_3(x) &= \frac{(1-x)^2}{128x}\left[4\, L(x) - 4x - 1 - (1-x)(1+3x)\frac{\ln(1-x)}{x}\right] \end{aligned} \tag{15}$$

where

$$L(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^2} = -\int_0^x \ln(1-t)\frac{\mathrm{d}t}{t}\,.$$

For arbitrary p this function is given by the sum (14) where each term is calculated as follows.

Theorem 2. For $p = 2, 3, \ldots$ and $m = 0, 1, \ldots$

$$S_m(x;\, p) = \left(-\frac{1}{4}\right)^p \sum_{k=0}^{p-1} c_{p-1-k}(m, x)\left[a_k(m, x) + (-1)^k b_{k-m}(m, x)\right] \tag{16}$$

where the functions c_l are given by the recurrence relations

$$\begin{aligned} &c_0(m, x) = 1 \\ &c_k(m, x) = \frac{m!}{k!(m+k)!} x^k - \sum_{l=0}^{k-1} \frac{m!}{(k-l)!(k-l+m)!} c_l(m, x) \qquad k \geqslant 1\,. \end{aligned} \tag{17}$$

The functions a_k are defined by

$$a_k(m,x) = (1-\delta_{m0}) \sum_{s=0}^{\min(k,m-1)} \frac{(-1)^s (m-s-1)!}{s!(k-s)!(k-s+m)!} \left[x^{k-s+m} - x^s\right] \tag{18}$$

and

$$b_s(m,x) = 0 \qquad s < 0 \tag{19}$$

$$b_s(m,x) = (-1)^s \, x^m \sum_{k=0}^{s} \frac{x^k \ln x + h_{mk}\left(x^{s-k} - x^k\right)}{k!(k+m)!(s+m-k)!(s-k)!} \qquad s \geqslant 0 \tag{20}$$

where

$$h_{mk} = \sum_{l=1}^{m+k} \frac{1}{l} + \sum_{l=1}^{k} \frac{1}{l}\,.$$

In section 4 we give explicit expressions for the first few components of the sum (16).

Equations (12), (14) and (16) provide a direct way of evaluating the main correction to zeta functions of almost circular domains. Let us consider a typical example.

Let the boundary $\partial\Omega$ of Ω be described in the polar coordinates $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ by the equation

$$r = 1 + \alpha\varphi(\theta) \qquad \alpha \ll 1\,. \tag{21}$$

Then the mapping $\omega : D \to \Omega$ to leading order in α is given by

$$\omega(z) = z + \alpha \frac{z}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z}\, \varphi(\theta)\, \mathrm{d}\theta + \mathrm{o}(\alpha)\,. \tag{22}$$

This representation is obtained in [18] via a perturbation expansion for the Green function. It allows one to calculate the leading term of the function $\epsilon(z)$ of (6), and the leading correction (12) to the zeta function for any given $\varphi(\theta)$ describing the boundary. For instance, if $\varphi(\theta)$ is a Fourier series

$$\varphi(\theta) = \sum_{k=-M}^{M} \varphi_k\, \mathrm{e}^{\mathrm{i}k\theta} \qquad \varphi_{-k} = \varphi_k^*$$

then equation (22) yields

$$\omega(z) = z + \alpha z \left[\varphi_0 + 2\sum_{k=1}^{M} \varphi_k\, z^k\right] + \mathrm{o}(\alpha)$$

so that

$$\epsilon(z) = 2\alpha \left[\varphi_0 + \sum_{k=1}^{M} (k+1)\,\mathrm{Re}(z^k \varphi_k)\right]\,.$$

Calculating the integral (13) yields $\epsilon_0(x) = 2\alpha\varphi_0$. Thus, according to (7) and (11) we have

$$\zeta(p;\Omega) = (1 + 2\alpha p\varphi_0)\zeta(p;D) + \mathrm{O}(\alpha^2)\,. \tag{23}$$

This very simple formula describes the leading correction to the zeta function due to a small perturbation (21) of the circular boundary.

A particular example is nearly circular ellipse of half axes $1+\alpha$ and 1:

$$x^2 + (1+\alpha)^2\, y^2 \leqslant (1+\alpha)^2\,.$$

Its boundary is described by (21) with $\varphi(\theta) = \cos^2\theta$, so that $\varphi_0 = \frac{1}{2}$ and

$$\zeta(p;\text{ellipse}) = (1 + \alpha p)\zeta(p;D) + \mathrm{O}(\alpha^2)\,.$$

The asymptotics (23) are valid for billiards with smooth boundaries. If the boundary has cusps, vertices, etc, the function $\varphi(\theta)$ and the mapping function $\omega(z)$ have singularities and the above approach should be modified. An important instance is the billiard on a regular n-sided polygon which we now proceed to study.

3. Regular n-sided polygons

Consider the Dirichlet problem (2) where Ω is a regular n-sided polygon P_n ($n = 3, 4, \ldots$) inscribed within the unit circle, so that the vertices of P_n are the roots of unity $\exp(2\pi l/n)$, $l = 0, 1, \ldots, n-1$.

As in the limit $n \to \infty$ P_n tends to the unit disc, it is natural to expect that $\zeta(p; P_n) \to \zeta(p; D)$. However, the asymptotics (23) are not valid, because the polar equation of the boundary of P_n cannot be expressed in the form (21) with a smooth function $\varphi(\theta)$.

We begin investigating $\zeta(p; P_n)$ along the same way as in the previous section. The mapping function $\omega : D \to P_n$ is well known:

$$\omega(z) = \gamma_n \int_0^z (1 - t^n)^{-2/n} \, \mathrm{d}t \tag{24}$$

where γ_n is fixed by the condition $\omega(1) = 1$ (one of the vertices of P_n is at $z = 1$):

$$\gamma_n = \left[\int_0^1 (1 - t^n)^{-2/n} \, \mathrm{d}t \right]^{-1} = \frac{\Gamma(1 - 1/n)}{\Gamma(1 + 1/n)\Gamma(1 - 2/n)} \, . \tag{25}$$

Thus

$$\left| \frac{\mathrm{d}\omega}{\mathrm{d}z} \right|^2 = \gamma_n^2 \left| 1 - z^n \right|^{-4/n} = \gamma_n^2 \left[1 - 2r^n \cos n\theta + r^{2n} \right]^{-2/n} \qquad z = r\mathrm{e}^{\mathrm{i}\theta} \, .$$

This expression can be rewritten by making use of the generating function for the Gegenbauer polynomials

$$(1 - 2xz + z^2)^{-\nu} = \sum_{k=0}^{\infty} C_k^{(\nu)}(z) \, x^k$$

in the form

$$\left| \frac{\mathrm{d}\omega}{\mathrm{d}z} \right|^2 = \gamma_n^2 \left[1 + \epsilon(z) \right] \tag{26}$$

where

$$\epsilon(z) = \sum_{k=1}^{\infty} C_k^{(2/n)}(\cos n\theta) \, r^{nk} \, . \tag{27}$$

As $n \to \infty$, this function is exponentially small everywhere inside the unit disc, $|\epsilon(z)| \propto r^{-n}$, $r < 1$, and can be treated as a perturbarion. Then equation (5) yields

$$\zeta(p; P_n) = \gamma_n^{2p} \, \mathrm{tr}[(1 + \epsilon) \, G_D]^p = \gamma_n^{2p} \left[\zeta(p; D) + p \, \mathrm{tr}(\epsilon \, G_D^p) + \mathrm{O}(\epsilon^2) \right] .$$

The term $\mathrm{tr}(\epsilon \, G_D^p)$ is described by (12) with

$$\epsilon_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \sum_{k=1}^{\infty} C_k^{(2/n)}(\cos n\theta) \, x^{nk/2} = \sum_{s=1}^{\infty} \frac{(2/n)_s^2}{(s!)^2} \, x^{ns} \tag{28}$$

where

$$(a)_0 = 1 \qquad (a)_s = a(a+1) \cdots (a+s-1) \qquad \text{for} \quad s \geqslant 1$$

and use has been made of the Fourier expansion for the Gegenbauer polynomials [19]

$$C_k^{(\nu)}(\cos\theta) = \sum_{l=0}^{k} \frac{(\nu)_l(\nu)_{k-l}}{l!(k-l)!}\, \exp\{-\mathrm{i}(k-2l)\theta\}\,. \tag{29}$$

Clearly, $\epsilon_0(x)$ is exponentially small as $n \to \infty$ for $x < 1$ while being $\propto n^{-2}$ at $x = 1$:

$$\epsilon_0(1) = \sum_{s=1}^{\infty} \frac{(2/n)_s^2}{(s!)^2} \sim \frac{4}{n^2} \sum_{s=1}^{\infty} \frac{1}{s^2} \qquad \text{as} \quad n \to \infty\,.$$

Hence, main contribution to the integral (12) comes from a small vicinity of the upper limit $x = 1$ which corresponds to the boundary of the unit disc where the mapping (24) has singularities at vertices of P_n. That is why one can replace the function g_p by its leading term as $x \to 1$. Expanding $J_m(j_{mk}\sqrt{x})$ in (14) at $x = 1$ gives

$$\begin{aligned} g_p(x) &= \frac{1}{4}\,(1-x)^2 \sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty} j_{|m|k}^{2(1-p)}\,(1+\mathrm{o}(1)) \\ &= \frac{1}{4}\,(1-x)^2\,\zeta(p-1;\,D)\,(1+\mathrm{o}(1))\,. \end{aligned} \tag{30}$$

This representation is valid only for $p \geqslant 3$. For $p = 2$ it breaks down, since $\zeta(z;\,D)$ has a pole at $z = 1$. That is why this case needs a special treatment and will be dealt with later.

Upon making use of equations (28) and (30) in (12), for $p \geqslant 3$ we have

$$\begin{aligned} \mathrm{tr}(\epsilon\, G_D^p) &= \int_0^1 \epsilon_0(x)\, g_p(x)\, \mathrm{d}x \\ &\sim \frac{1}{4}\,\zeta(p-1;\,D) \sum_{s=1}^{\infty} \frac{(2/n)_s^2}{(s!)^2} \int_0^1 x^{ns}(1-x)^2\, \mathrm{d}x \\ &= \frac{1}{2}\,\zeta(p-1;\,D) \sum_{s=1}^{\infty} \frac{(2/n)_s^2}{(s!)^2(ns+1)(ns+2)(ns+3)}\,. \end{aligned}$$

Using the asymptotics ($n \to \infty$)

$$(2/n)_s \sim \frac{2}{n}\,(s-1)! \qquad \text{for} \quad s \geqslant 1 \tag{31}$$

we get

$$\mathrm{tr}(\epsilon\, G_D^p) = \frac{2}{n^5}\,\zeta(5)\,\zeta(p-1;\,D) + \mathrm{o}(n^{-5})$$

where $\zeta(m)$ is the Riemann zeta function, $\zeta(m) = \sum_{k=1}^{\infty} k^{-m}$. Thus, we arrive at

Theorem 3. For $p = 2, 3, \ldots$ the zeta function $\zeta(p;\,P_n)$ has the following asymptotics as $n \to \infty$:

$$\zeta(p;\,P_n) = \gamma_n^{2p}\left\{\zeta(p;\,D) + \frac{2p}{n^5}\,\zeta(5)\,\zeta(p-1;\,D) + \mathrm{o}(n^{-5})\right\} \tag{32}$$

where γ_n is given by (25).

These asymptotics are not valid for $p = 2$. In this case the result is as follows.

Theorem 4. The asymptotics of the zeta function $\zeta(2; P_n)$ in the limit $n \to \infty$ are of the form

$$\zeta(2; P_n) = \gamma_n^4 \left\{ \zeta(2; D) + 2\zeta(5)\frac{\ln n}{n^5} + \frac{d}{n^5} + \mathrm{o}(n^{-5}) \right\} \tag{33}$$

where

$$d = \left(2\gamma - 3 + \frac{16}{405}\right)\zeta(5) + 2\sum_{k=2}^{\infty}\frac{\ln k}{k^5} \tag{34}$$

and γ is Euler's constant, $\gamma = 0.5772\ldots$.

We prove this theorem in section 5. Note that the logarithmic term of (33) is directly related to the logarithmic singularity at $x = 1$ of the corresponding function g_2 in (15). For $p \geqslant 3$ behaviour of $g_p(x)$ as $x \to 1$, described by (30), is smoother, for the series (14) converge faster. That is why in this case the asymptotics (32) do not have a logarithmic correction.

Note that as $n \to \infty$

$$\gamma_n = 1 - \frac{\pi^2}{3n^2} + \mathrm{O}(n^{-3})$$

so that main correction to the zeta function of the circular billiard in (32) and (33) comes from the normalization factor γ_n of the mapping (24).

The asymptotics (32) and (33) can be used to estimate zeta functions of regular n-sided polygons with $n \geqslant 5$ for which there are no exact formulae. The first-order approximation is

$$\zeta^{(0)}(p; P_n) = \gamma_n^{2p}\,\zeta(p; D) \qquad p = 2, 3, 4, \ldots \tag{35}$$

where all corrections $\propto n^{-5}$ in (32) and (33) are neglected. The second approximation takes them into account. For instance, for $p = 2$ it is

$$\zeta^{(1)}(2; P_n) = \gamma_n^4 \left\{ \zeta(2; D) + 2\zeta(5)\, n^{-5}\ln n + \frac{d}{n^5} \right\} \tag{36}$$

where d is defined in (34).

In order to check how reasonable these approximations may be expected to be for finite n, let us compare them with known exact formulae for an equilateral triangle and a square. They were obtained in [2] by a direct evaluation making use of explicit expressions for the eigenvalues of the corresponding Dirichlet problems:

$$\zeta(p; P_n) = \left(\frac{a}{\pi}\alpha_n\right)^{2p} \left[\zeta(p)L_n(p) - \zeta(2p)\right] \qquad n = 3, 4 \tag{37}$$

where a is the side length, $\zeta(m)$ is the Riemann zeta function, $\alpha_3 = \frac{3}{4}, \alpha_4 = 1$ and

$$L_3(p) = \sum_{m=0}^{\infty}\left[\frac{1}{(3m+1)^p} - \frac{1}{(3m+2)^p}\right]$$

$$L_4(p) = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^p}\,.$$

In our case, the side lengths are fixed by the condition that the triangle and square are inscribed within the unit circle, so that $a = \sqrt{3}$ for P_3 and $a = \sqrt{2}$ for P_4.

Table 1 gives results for $\zeta(2; P_n)$ obtained by the approximate formulae (35), (36) and compares them with the exact values for $n = 3, 4$. Surprisingly, both approximations, and

Table 1. Zeta function $\zeta(2; P_n)$ of regular n-sided polygons. Exact values for $n = 3, 4$ are given by (37). The approximations $\zeta^{(0)}$ and $\zeta^{(1)}$ are defined by (35) and (36). The $n = \infty$ entry is the value of $\zeta(2; D)$ from (10).

n	Exact (10^{-2})	$\zeta^{(1)}$ (10^{-2})	$\zeta^{(0)}$ (10^{-2})	γ_n
3	0.593 06...	0.5264	0.5068	0.5661
4	1.742 67...	1.7061	1.6710	0.7628
5		2.6204	2.5948	0.8515
6		3.2340	3.2180	0.8985
7		3.6459	3.6361	0.9264
8		3.9300	3.9240	0.9442
9		4.1322	4.1284	0.9563
10		4.2804	4.2779	0.9648
11		4.3918	4.3901	0.9711
12		4.4775	4.4763	0.9758
13		4.5447	4.5438	0.9795
14		4.5983	4.5976	0.9824
15		4.6417	4.6412	0.9847
16		4.6773	4.6770	0.9866
17		4.7069	4.7066	0.9881
18		4.7317	4.7315	0.9895
19		4.7528	4.7526	0.9906
20		4.7707	4.7706	0.9915
∞	4.936 68...			1

especially $\zeta^{(1)}$, work fairly well even for the lowest n, being already within 10% of the exact value for the triangle. The agreement rather improves for the square. Thus, one may expect that the approximations (32) and (36) give rather accurate estimates for the zeta functions $\zeta(p; P_n)$ of a regular n-sided polygon with $n \geqslant 5$ where no exact formulae are known.

We also note that the difference between the two approximations (due to the terms $\propto n^{-5}$ in (36)) decreases very fast with increase of n, and is very small already for $n = 6$. The main difference between $\zeta(p; P_n)$ at finite n and its limiting value $\zeta(p; D)$ is due to the scaling factor γ_n of the mapping (24). As can be seen from table 1, $\zeta(p; P_n)$ reaches the limit very slowly. This is due to the slow convergence $\gamma_n \to 1$.

4. New addition theorems for Bessel functions

In this section we study the sums (14) and obtain equations (15) and other related results.

The starting point of our approach resembles the derivation of the Kneser–Sommerfeld expansion for Bessel functions [9]. For $0 \leqslant x \leqslant 1$ and $m \geqslant 0$, consider the function

$$\begin{aligned} f_m(z, x) &= H_m^{(1)}(xz) H_m^{(2)}(z) - H_m^{(1)}(z) H_m^{(2)}(xz) \\ &= 2\mathrm{i}\,[J_m(z) Y_m(xz) - J_m(xz) Y_m(z)] \end{aligned} \tag{38}$$

and the integral

$$I = \frac{1}{2\pi\mathrm{i}} \oint_{C_R} f_m(z, x)\, \frac{J_m(xz)}{J_m(z)}\, \frac{\mathrm{d}z}{z^{2p-1}} \qquad p = 2, 3, \ldots \tag{39}$$

over a circle of radius R around the origin. Using the relations ($\lambda > 0$)

$$J_m\left(\lambda \mathrm{e}^{\pm\mathrm{i}\pi}\right) = (-1)^m J_m(\lambda) \qquad Y_m\left(\lambda \mathrm{e}^{\pm\mathrm{i}\pi}\right) = (-1)^m [Y_m(\lambda) \pm 2\mathrm{i} J_m(\lambda)]$$

it can readily be seen that $f_m\left(\lambda e^{\pm i\pi}, x\right) = f_m(\lambda, x)$ for $\lambda > 0$, so that $f_m(z, x)$ is an analytic function of z on the whole z plane. Thereby, the function $f_m(z, x)J_m(xz)/J_m(z)$ is bounded on C_R when $0 \leqslant x \leqslant 1$, so that the integral (39) vanishes as $R \to \infty$. The integrand has simple poles at $z = \pm j_{mk}$ and a pole at $z = 0$. As the integrand of (39) is an even function of z, we get

$$2\sum_{k=1}^{\infty} \operatorname{Res}\left[\frac{f_m(z, x)J_m(xz)}{z^{2p-1}J_m(z)}; z = j_{mk}\right] + \operatorname{Res}\left[\frac{f_m(z, x)J_m(xz)}{z^{2p-1}J_m(z)}; z = 0\right] = 0\,. \qquad (40)$$

Equation (38) yields

$$\begin{aligned} f_m\left(j_{mk}, x\right) &= -2\mathrm{i}J_m\left(j_{mk}x\right)Y_m\left(j_{mk}\right) \\ &= -2\mathrm{i}\frac{J_m\left(j_{mk}x\right)}{J_m'\left(j_{mk}\right)}\left[Y_m\left(j_{mk}\right)J_m'\left(j_{mk}\right) - J_m\left(j_{mk}\right)Y_m'\left(j_{mk}\right)\right] \\ &= \frac{4\mathrm{i}J_m\left(j_{mk}x\right)}{\pi j_{mk}J_m'\left(j_{mk}\right)} \end{aligned}$$

where the expression in square brackets is the Wronskian

$$W\left\{Y_m(x), J_m(x)\right\} = -2/(\pi x)\,.$$

Therefore

$$\operatorname{Res}\left[\frac{f_m(z, x)J_m(xz)}{z^{2p-1}J_m(z)}; z = j_{mk}\right] = \frac{4i\ J_m^2\left(j_{mk}x\right)}{j_{mk}^{2p}\left[J_m'\left(j_{mk}\right)\right]^2}$$

and equation (40) yields

$$S_m(x^2; p) \equiv \sum_{k=1}^{\infty}\frac{J_m^2\left(j_{mk}x\right)}{j_{mk}^{2p}\left[J_m'\left(j_{mk}\right)\right]^2} = \frac{\pi\mathrm{i}}{8}\operatorname{Res}\left[\frac{f_m(z, x)J_m(xz)}{z^{2p-1}J_m(z)}; z = 0\right]. \qquad (41)$$

This equation provides a direct way to calculate the sums S_m for any integer $p \geqslant 1$.

We now proceed to evaluate the residue in (41). Consider first the function $f_m(z, x)$ as $z \to 0$. Upon substituting into (38) the expansions of the Bessel functions in powers of z, a straightforward calculation yields

$$f_m(z, x) = \frac{2\mathrm{i}}{\pi}x^{-m}\sum_{k=0}^{\infty}\left(\frac{z^2}{4}\right)^k\left[(-1)^k a_k(m, x^2) + b_{k-m}(m, x^2)\right] \qquad (42)$$

where the coefficients a_k and b_k are defined in (18) and (19). A similar calculation gives

$$\frac{J_m(xz)}{J_m(z)} = x^m\sum_{k=0}^{\infty}\left(-\frac{z^2}{4}\right)^k c_k(m, x^2) \qquad (43)$$

where the coefficients c_k are defined by the recurrence (19). To complete evaluating the residue in (41), we have to calculate the term $\propto z^{2p-2}$ of the product of the series (42) and (43). This gives equation (16) for sums $S_m(x; p)$ which can be regarded as a set of new addition theorems for Bessel functions.

Explicit expressions for the first few coefficients of the sum (16) can readily be obtained:

$$c_0(m, x) = 1$$

$$c_1(m, x) = \frac{x-1}{m+1}$$

$$c_2(m,x) = \frac{x-1}{2(m+1)(m+2)}\left[x-1-\frac{2}{m+1}\right]$$

$$\vdots$$

$$a_0(m,x) = \frac{1-\delta_{m0}}{m}(x^m-1)$$

$$a_1(m,x) = \frac{1-\delta_{m0}}{m}\left[\frac{x^{m+1}-1}{m+1} - (1-\delta_{m1})\frac{x^m-x}{m-1}\right]$$

$$a_2(m,x) = \frac{1-\delta_{m0}}{2m}\left[\frac{x^{m+2}}{(m+1)(m+2)} + 2(1-\delta_{m1})\frac{x-x^{2m+1}}{m^2-1}\right.$$
$$\left. + \frac{(1-\delta_{m1})(1-\delta_{m2})}{(m-1)(m-2)}(x^m-x^2)\right]$$

$$\vdots$$

$$b_0(m,x) = \frac{x^m \ln x}{(m!)^2}$$

$$b_1(m,x) = -\frac{x^m}{m!(m+1)!}\left[(1+x)\ln x + \frac{m+2}{m+1}(1-x)\right]$$

$$b_2(m,x) = \frac{x^m}{m!(m+2)!}\left\{\left[\frac{1}{2}(1+x)^2 + \frac{x}{m+1}\right]\ln x + \frac{1-x^2}{2}\left[\frac{1}{m+1}+\frac{1}{m+2}+\frac{3}{2}\right]\right\}$$

$$\vdots$$

Substituting these expressions in (16) leads to the following results for the sums $S_m(x;\,p)$ with $p = 1, 2, 3$:
(i) $p = 1$:

$$\sum_{k=1}^{\infty} \frac{J_m^2\left(j_{mk}\sqrt{x}\right)}{j_{mk}^2\left[J_m'\left(j_{mk}\right)\right]^2} = -\frac{1}{4}\ln x \qquad m = 0$$
$$= \frac{1-x^m}{4m} \qquad m = 1, 2, \ldots\,.$$

(ii) $p = 2$:

$$\sum_{k=1}^{\infty} \frac{J_m^2\left(j_{mk}\sqrt{x}\right)}{j_{mk}^4\left[J_m'\left(j_{mk}\right)\right]^2} = \frac{1}{8}[1-x+x\ln x] \qquad m = 0$$
$$= \frac{1}{16}[x(x-1)-x\ln x] \qquad m = 1$$
$$= \frac{1}{8(m+1)}\left[\frac{x^{m+1}}{m} - \frac{x^m}{m-1} + \frac{x}{m(m-1)}\right] \qquad m = 2, 3, \ldots\,.$$

(iii) $p = 3$:

$$\sum_{k=1}^{\infty} \frac{J_m^2\left(j_{mk}\sqrt{x}\right)}{j_{mk}^6\left[J_m'\left(j_{mk}\right)\right]^2} = -\frac{1}{256}[6x^2\ln x + (1-x)(11x-5)] \qquad m = 0$$

$$= \frac{x}{256}\left[4x\ln x + \frac{1}{3}(1-x)(5x+7)\right] \qquad m = 1$$

$$= -\frac{x^2}{256}\left[\ln x + \frac{1}{18}(1-x)(25-7x)\right] \qquad m = 2$$

$$= -\frac{x^m}{64m(m+1)^2}\left[\frac{2m+3}{m+2}x^2 - \frac{4m}{m-1}x + \frac{m(2m+5)}{m^2-4}\right]$$

$$+\frac{3x^2}{32m(m^2-1)(m^2-4)} \qquad m = 3, 4, \ldots$$

These formulae are the first members of the infinite set of addition theorems for the Bessel functions generated by equation (16).

Substituting the above expressions for $p = 2, 3$ into the definition (14) of the functions g_p yields equations (15). For $p = 1$, the sum over m in (14) diverges. This is another illustration of singularity of the zeta function $\zeta(p; D)$ at $p = 1$.

5. The asymptotics of $\zeta(2; P_n)$

In this section we prove equation (33). When $p = 2$, equations (5) and (26) yield

$$\zeta(2; P_n) = \gamma_n^4\left\{\zeta(2; D) + 2\,\mathrm{tr}(\epsilon\, G_D^2) + \mathrm{tr}(\epsilon\, G_D\, \epsilon\, G_D)\right\}. \tag{44}$$

First, we calculate the term $\propto \epsilon$ by making use of equations (12), (28) and (15):

$$\mathrm{tr}(\epsilon\, G_D^2) = \int_0^1 \epsilon_0(x)\, g_2(x)\, \mathrm{d}x = \frac{1}{8}\sum_{s=1}^{\infty}\frac{(2/n)_s^2}{(s!)^2}\, I(ns)$$

where

$$I(k) = -\int_0^1 x^{k-1}(1-x)^2\ln(1-x)\,\mathrm{d}x$$

$$= \sum_{l=1}^{\infty}\frac{1}{l}\int_0^1 x^{k-1+l}(1-x)^2\,\mathrm{d}x$$

$$= \sum_{l=1}^{\infty}\frac{1}{l}\left[\frac{1}{k+l} - \frac{2}{k+l+1} + \frac{1}{k+l+2}\right]$$

$$= \frac{2}{k(k+1)(k+2)}\sum_{l=1}^{k}\frac{1}{l} - \frac{2}{(k+1)^2} + \frac{1}{(k+2)^2} + \frac{1}{(k+1)(k+2)}.$$

This yields the exact representation

$$\mathrm{tr}(\epsilon\, G_D^2) = \frac{1}{4n}\sum_{s=1}^{\infty}\frac{(2/n)_s^2}{(s!)^2 s(ns+1)(ns+2)}\left\{\sum_{l=1}^{ns}\frac{1}{l} - \frac{3}{2} + \frac{1}{ns+1} + \frac{1}{ns+2}\right\}.$$

Using the asymptotics (31) and

$$\sum_{l=1}^{k}\frac{1}{l}=\gamma+\ln k+\frac{1}{2k}-\frac{1}{12k(k+1)}+\cdots$$

gives

$$\mathrm{tr}(\epsilon\, G_D^2)=\frac{1}{n^5}\left[\zeta(5)\ln n+\left(\gamma-\frac{3}{2}\right)\zeta(5)+\sum_{s=1}^{\infty}\frac{\ln s}{s^5}\right]+\mathrm{O}(n^{-6})\,. \qquad (45)$$

This describes the second term of (44) in the limit $n\to\infty$.

We now proceed to evaluate the last term of (44)

$$\mathrm{tr}(\epsilon\, G_D\,\epsilon\, G_D)=\int_{D\times D}\mathrm{d}^2z_1\,\mathrm{d}^2z_2\,\epsilon(z_1)\epsilon(z_2)\,G_D^2(z_1,z_2)\,. \qquad (46)$$

As follows from equation (28), $\epsilon(z)$ is exponentially small as $n\to\infty$ for $|z|<1$ but vanish only as a negative power of n on the boundary $|z|=1$. Hence, main contribution to the integral (46) as $n\to\infty$ comes from a small vicinity of the boundaries $|z_1|=1$ and $|z_2|=1$. Therefore, we can replace the Green function in (46) by its leading term when both arguments are close to the boundary. As follows from (4), when $|z_{1,2}|\to 1$:

$$G_D(z_1,z_2)\sim\frac{\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)}{4\pi\left|1-z_1^*z_2\right|^2}\,.$$

Upon writing this in polar coordinates $z_k=r_k\mathrm{e}^{\mathrm{i}\theta_k},\ k=1,2$ and making use of the Fourier expansion

$$\frac{(1-x^2)^2}{(1-2x\cos\theta+x^2)^2}=\sum_{m=0}^{\infty}G_m(x)\,\cos m\theta$$

where

$$G_0(x)=\frac{1+x^2}{1-x^2}$$

$$G_1(x)=\frac{4x}{1-x^2}$$

$$G_m(x)=2x^m\left[\frac{2}{1-x^2}+m-1\right]\qquad m\geqslant 2$$

we arrive at the representation

$$G_D^2(z_1,z_2)\sim\frac{1}{16\pi^2}\,W(r_1,r_2)\sum_{m=0}^{\infty}G_m(r_1r_2)\,\cos m(\theta_1-\theta_2)$$

where

$$W(r_1,r_2)=\frac{(1-r_1^2)^2(1-r_2^2)^2}{1-r_1^2r_2^2}\,.$$

We substitute this representation and (27) into (46) and evaluate the resulting angular integrals by making use of

$$\int_0^{2\pi}C_k^{(\nu)}(\cos n\theta)\mathrm{e}^{\mathrm{i}m\theta}=\pi\sum_{l=0}^{k}\frac{(\nu)_l(\nu)_{k-l}}{l!(k-l)!}\left\{\delta_{n(k-2l),m}+\delta_{n(k-2l),-m}\right\}$$

which follows from (29). As a result, one gets

$$\mathrm{tr}(\epsilon\, G_D\, \epsilon\, G_D) \sim \int_0^1 \int_0^1 \mathrm{d}r_1\, \mathrm{d}r_2\; W(r_1, r_2) \sum_{s=0}^{\infty} G_{ns}(r_1 r_2)$$

$$\times \sum_{l_1=\delta_{s0}}^{\infty} \sum_{l_2=\delta_{s0}}^{\infty} r_1^{n(s+2l_1)+1} r_2^{n(s+2l_2)+1}\, \frac{(2/n)_{l_1}(2/n)_{l_1+s}(2/n)_{l_2}(2/n)_{l_2+s}}{l_1!(l_1+s)!l_2!(l_2+s)!} \tag{47}$$

$$= A(n) + B(n)$$

where A stands for the contribution of all terms of this sum with $s \geqslant 1$ and B for that with $s = 0$.

According to equation (31), main contribution to A comes from the terms with $l_1 = l_2 = 0$ and is given by

$$A(n) \sim \frac{1}{n^2} \int_0^1 \int_0^1 \mathrm{d}r_1\, \mathrm{d}r_2\; W(r_1, r_2) \sum_{s=1}^{\infty} \frac{1}{s^2} (r_1 r_2)^{2ns+1}\, G_{ns}(r_1 r_2) \tag{48}$$

while using (31) for the terms of (47) with $s = 0$ yields

$$B(n) \sim \frac{4}{n^2} \int_0^1 \int_0^1 \mathrm{d}r_1\, \mathrm{d}r_2\; W(r_1, r_2) G_0(r_1 r_2) \sum_{l_1=1}^{\infty} \frac{r_1^{nl_1}}{l_1^2} \sum_{l_2=1}^{\infty} \frac{r_2^{nl_2}}{l_2^2}\,. \tag{49}$$

Again, the leading contribution to these integrals comes from the endpoints $r_i = 1$. Introducing new variables $x_i = 1 - r_i^2$, $i = 1, 2$ and expanding the integrand of (48) at $x_i = 0$ yields

$$A(n) \sim \frac{1}{2n^2} \sum_{s=1}^{\infty} \frac{1}{s^2} \left\{ (ns - 1) I_2\left(\frac{3}{2}ns\right) + 2I_3\left(\frac{3}{2}ns\right) \right\} \tag{50}$$

where

$$I_m(\lambda) = \int_0^1 \int_0^1 \mathrm{d}x_1\, \mathrm{d}x_2\; \frac{x_1^2 x_2^2}{(x_1 + x_2)^m} \exp\{-\lambda(x_1 + x_2)\} \qquad m = 2, 3\,.$$

In the limit $\lambda \to \infty$ one gets

$$I_m(\lambda) = a_m\, \lambda^{m-6}\, [1 + \mathrm{o}(1)]$$

where

$$a_m = \int_0^\infty \int_0^\infty \mathrm{d}x_1\, \mathrm{d}x_2\; \frac{x_1^2 x_2^2}{(x_1 + x_2)^m} \exp\{-x_1 - x_2\}\,.$$

Upon introducing the variables $t_1 = x_1 + x_2$, $t_2 = x_1 - x_2$ this integral is evaluated explicitly to give

$$a_m = \int_0^\infty \mathrm{d}t_1\; t_1^{-m}\, \mathrm{e}^{-t_1} \int_0^{t_1} \left(\frac{t_1^2 - t_2^2}{4}\right)^2 \mathrm{d}t_2 = \frac{1}{30} \int_0^\infty \mathrm{d}t\; t^{5-m}\, \mathrm{e}^{-t}$$

$$= \tfrac{1}{30}\,(5 - m)!\,.$$

Then equation (50) yields

$$A(n) = \left(\frac{2}{3}\right)^4 \frac{a_2 + 3a_3}{2n^5}\, \zeta(5) + \mathrm{o}(n^{-5}) \qquad \text{as} \quad n \to \infty\,.$$

In a similar way one can show that the second term (49) is neglegible: $B(n) = \mathrm{O}(n^{-7})$ as $n \to \infty$. Therefore, we get the asymptotics

$$\mathrm{tr}(\epsilon\, G_D\, \epsilon\, G_D) = \frac{16\,\zeta(5)}{405\, n^5} + \mathrm{o}(n^{-5}) \qquad \text{as} \quad n \to \infty\,.$$

Combining this with equation (45) in (44) yields (33).

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